17. Let P and Q be two points in $\mathbb{H}$ which don't lie on same vertical line. Use your knowledge from high school and find intersection of the Euclidean perpendicular bisector of the Euclidean line segment from P to Q with x-axis.

$[\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1},\ k=\frac{y_2-y_1}{x_2-x_1},\ s: y=k_s x+n,\ k_s=-\frac{1}{k},\ s: y=-\frac{x_2-x_1}{y_2-y_1}x+\frac{y_2^2-y_1^2+x_2^2-x_1^2}{2(y_2-y_1)}]$

18. Let $P,Q\in\mathbb{H}$ denote two different points, and let $p(P,Q)={}_cL_r$ (points P and Q lie on Poincaré line ${}_cL_r$). Use your knowledge of Euclidean geometry to prove that c is the x-coordinate of the intersection of the Euclidean perpendicular bisector of the Euclidean line segment from P to Q with x-axis. $[P(x_1,y_1),\ Q(x_2,y_2),\ S$ midpoint of $\overline{PQ},\ R(c,0),\ \triangle PSR\cong\triangle QSR]$

Definition (incidence geometry)

An abstract geometry $\{\mathcal{S},\mathcal{L}\}$ is an incidence geometry if

(i) Every two distinct points in $\mathcal{S}$ lie on a unique line.

(ii) There exists a set of three non-collinear points.

Notation. If $\{\mathcal{S},\mathcal{L}\}$ is an incidence geometry and $P,Q\in\mathcal{S}$ then the unique line p on which both P and Q lie will be written $p=p(P,Q)$.

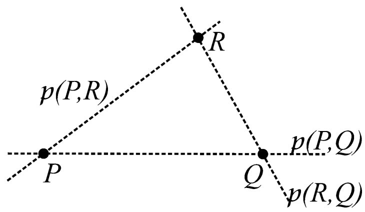
19. Show that the Cartesian Plane $\mathcal{C}=\{\mathbb{R}^2,\mathcal{L}_E\}$ is an incidence geometry.

$[1^\circ\ P,Q\in L_a,\ P,Q\in L_{a'};\ 2^\circ\ P,Q\in L_a,\ P,Q\in L_{m,b};\ 3^\circ\ P,Q\in L_{m,b},\ P,Q\in L_{n,c}]$

20. Show that the Poincaré Plane $\mathcal{H}=\{\mathbb{H},\mathcal{L}_H\}$ is an incidence geometry.

$[1^\circ\ P,Q\in {}_aL,\ P,Q\in {}_{a'}L;\ 2^\circ\ P,Q\in {}_aL,\ P,Q\in {}_cL_r;\ 3^\circ\ P,Q\in {}_cL_r,\ P,Q\in {}_dL_s]$

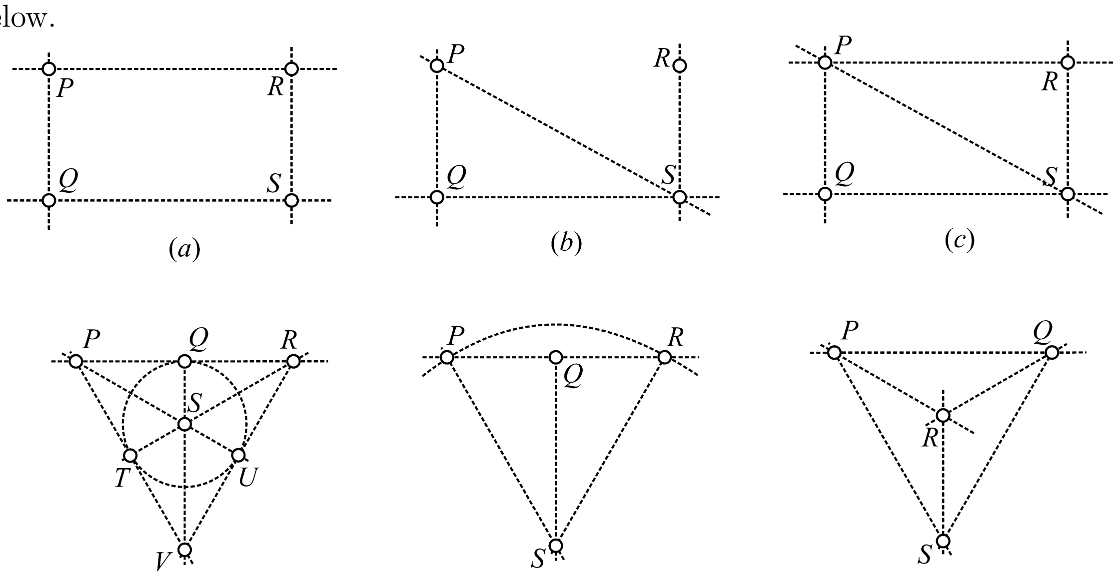
21. Let $\mathcal{S}=\{P,Q,R\}$ and $\mathcal{L}=\{p(P,Q),p(P,R),p(Q,R)\}$. Show that $\{\mathcal{S},\mathcal{L}\}$ is an incidence geometry. Note that this example has only finitely many (in fact, three) points. It may be pictured as in figure on right side. It is called the 3-point geometry. The dotted lines indicate which points lie on the same line.



[we need to show that two properties from definition of incidence geometry are satisfied]

22. Let $\mathcal{S}=\mathbb{R}^2-\{(0,0)\}$ and $\mathcal{L}$ be the set of all Cartesian lines which lie in $\mathcal{S}$. Show that $\{\mathcal{S},\mathcal{L}\}$ is not an incidence geometry. [property (i) from definition of incidence geometry is not satisfied]

23. Some finite geometries are defined pictorially (as in the 3-point geometry of Problem 21) by figure below.



(a) (b) (c) (d) (e) (f)

(i) In each example list the set of lines. $[(d)\ p(P,Q,R),\ p(P,T,V),\ p(P,S,U)...]$

(ii) Which of these geometries are abstract geometries? $[(d),\ (e),\ (f)]$

(iii) Which of these geometries are incidence geometries? $[(d),\ (f)]$

"Prove" may mean "find a counterexample".

24. Let $\{\mathcal{S}_1, \mathcal{L}_1\}$ and $\{\mathcal{S}_2, \mathcal{L}_2\}$ be abstract geometries. Let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ and $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$. Prove that $\{\mathcal{S}, \mathcal{L}\}$ is an abstract geometry. [$A \in \mathcal{S}_1$, $B \in \mathcal{S}_2$, $A, B \notin \mathcal{S}_1 \cap \mathcal{S}_2$]

25. Let $\{\mathcal{S}_1, \mathcal{L}_1\}$ and $\{\mathcal{S}_2, \mathcal{L}_2\}$ be abstract geometries. Let $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$ and $\mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2$. Prove that $\{\mathcal{S}, \mathcal{L}\}$ is an abstract geometry. [look at figures (e) and (f) from Problem 23]

Definition (parallel lines)

If ℓ_1 and ℓ_2 are lines in an abstract geometry then ℓ_1 is parallel to ℓ_2 (written $\ell_1 \parallel \ell_2$) if either $\ell_1 = \ell_2$ or $\ell_1 \cap \ell_2 = \emptyset$.

26. Let ℓ_1 and ℓ_2 be lines in an incidence geometry. Show that if $\ell_1 \cap \ell_2$ has two or more points then it $\ell_1 = \ell_2$. [$P \in \ell_1 \wedge Q \in \ell_1 \Rightarrow \ell_1 = p(P,Q)$, $P \in \ell_2 \wedge Q \in \ell_2 \Rightarrow \ell_2 = p(P,Q)$]

27. Find all lines through $P(0,1)$ which are parallel to the vertical line L_6 in the Cartesian Plane $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$. [$L_0$, $L_6 \cap L_{k,1} = \emptyset$]

28. Find all lines in the Poincaré Plane $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ through $P(0,1)$ which are parallel to the type I line ${}_6L$. [${}_aL \Rightarrow a = 0$, ${}_cL_r \Rightarrow c^2 + 1^2 = r^2$, $c < 35/12$; GeoGebra: $r = \text{sqrt}(c\char`^2 + 1)$, $y = \text{sqrt}(r\char`^2 - (x-c)\char`^2)$]

29. Find all lines of Riemann Sphere $\mathcal{R} = \{S^2, \mathcal{L}_R\}$ through $N(0,0,1)$ which are parallel to the spherical line (great circle) $\mathcal{G} = \{(x,y,z) \in S^2 \mid z = 0\}$. [$c = 0$, $ax + by = 0$, such line does not exist]

Definition (equivalence relation)

An equivalence relation on a set X is a relation $R \subseteq X \times X$ such that

(i) $(x,x) \in R$ for all $x \in X$ (reflexive property),

(ii) $(x,y) \in R$ implies $(y,x) \in R$ (symmetric property),

(iii) $(x,y) \in R$ and $(y,z) \in R$ imply $(x,z) \in R$ (transitive property).

Given an equivalence relation R on a set X, we usually write $x \sim y$ instead of $(x,y) \in R$. If the equivalence relation already has an associated notation such as $=$, $\equiv$, or $\cong$, we will use that notation.

30. Let $\{\mathcal{S}, \mathcal{L}\}$ be an abstract geometry. If ℓ_1 and ℓ_2 are lines in $\mathcal{L}$ we write $\ell_1 \sim \ell_2$ if ℓ_1 is parallel to ℓ_2. Prove that $\sim$ is an equivalence relation. If $\{\mathcal{S}, \mathcal{L}\}$ is the Cartesian Plane then each equivalence class can be characterized by a real number or infinity. What is this number?

[$\ell_1 \cap \ell_2 = \emptyset$, $\ell_2 \cap \ell_3 = \emptyset$, $p(P,R) \parallel p(S,Q)$, $p(P,R) \parallel p(T,Q)$, $\sim$ is not an equivalence relation]

31. There is a finite geometry with 7 points such that each line has exactly 3 points on it. Find this geometry. How many lines are there? [Fano plane]

32. Let $\mathcal{S} = \mathbb{R}^2$ and, for a given choice of a, b, and c, let

$$J_{a,b,c} = \{(x,y) \in \mathbb{R}^2 \mid ax + by = c\}.$$

Let $\mathcal{L}_J$ be the set of all $J_{a,b,c}$ with at least one of a and b nonzero. Prove that $\{\mathbb{R}^2, \mathcal{L}_J\}$ is an incidence geometry. (Note that this incidence geometry gives the same family of lines as the Cartesian Plane. The point here is that there are different ways to describe the set of lines of this geometry.)

[$1°\, x_1 = x_2$, $2°\, y_1 = y_2$, $3°\, x_1 \neq x_2$ and $y_1 \neq y_2$]

33. Define a relation $\sim$ on S^2 as follows. If $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ then $A \sim B$ if either $A = B$ or $A = -B = (-x_2, -y_2, -z_2)$. Prove $\sim$ is an equivalence relation.

34. Let $\{\mathcal{S}, \mathcal{L}\}$ be an abstract geometry and assume that $\mathcal{S}_1 \subseteq \mathcal{S}$. We define an $\mathcal{S}_1$-line to be any subset of $\mathcal{S}_1$ of the form $\ell \cap \mathcal{S}_1$ where ℓ is a line of $\mathcal{S}$ and where $\ell \cap \mathcal{S}_1$ has at least two points. Let $\mathcal{L}_1$ be the collection of all $\mathcal{S}_1$-lines. Prove that $\{\mathcal{S}_1, \mathcal{L}_1\}$ is an abstract geometry. $\{\mathcal{S}_1, \mathcal{L}_1\}$ is called the geometry induced from $\{\mathcal{S}, \mathcal{L}\}$.